COMMUNICATING MATHEMATICAL REASONING: MORE THAN JUST TALK

Peter Gould

NSW Department of Education and Training, Australia

Communicating is one of the five processes contributing to Working Mathematically in the New South Wales elementary curriculum. Students need to learn to use appropriate language and representations to formulate and express mathematical ideas in written, oral and diagrammatic form. A research lesson on shaping and communicating mathematical ideas arising from a curious relationship between the difference between the digits in a subtraction and the answer is used in a composite Year 5–6 class. The lesson study highlighted the need to establish and monitor a common ground for effective mathematical communication to lead to developing shared meaning in a classroom. The need to introduce an appropriate way to notate the number relationship being described orally became apparent in reflecting on the study lesson.

PATTERNS OF COMMUNICATION

The curriculum process strand of Working Mathematically in the New South Wales mathematics syllabus draws on ways of seeing, questioning, interpreting, reasoning and communicating. In contrast to curriculum content strands such as number and measurement, teacher knowledge of what constitutes effective mathematical communication has rarely been explicitly addressed in classrooms in Australia. Although research into the social construction of knowledge (e.g., Cobb & Bauersfeld, 1995) and the teaching practices that contribute to normative patterns of interaction and discourse in mathematics classrooms (Wood, 1994; Wood, Williams, & McNeal, 2006) is well documented, typically research on mathematical communication focuses on analysing classroom discourse. Current socio-cultural perspectives on mathematics learning call on teachers to reinvent themselves in ways that facilitate student learning rather than transmit knowledge. For teachers, this means adopting new roles, and acquiring a new repertoire of teacher talk. This lesson study is designed to investigate the necessary components of classroom discourse or "maths talk" to enable students to communicate their reasoning about place value.

Perhaps the best-known and most ubiquitous pattern of exchange in classrooms is the Initiate-Response-Evaluate (IRE) cycle (Hoetker & Ahlbrant, 1969; Mehan, 1979; Sinclair & Colthard, 1975). In this pattern of exchange, the teacher initiates the exchange by asking a question about a known fact or idea, a student replies and the teacher evaluates the response as to whether it is correct. The limitations of this pattern of exchange are well known. A less limiting form of mathematical communication than IRE is described by Elicitation-Response-Elaboration (ERE). The practice of the teacher eliciting a response, the student responding and the teacher elaborating can be expanded through several linked cycles into a whole class approach to mathematical communication.

The teacher's role, which is critical to developing effective communication skills in students, can be described as having three connected domains: supporting students in making contributions, establishing and monitoring a common ground and, guiding the mathematics (Staples, 2007). Within this lesson study, particular attention is paid to eliciting ideas, scaffolding the production of ideas through providing structure, revoicing and extending, all of which fall under Staples' domain of *supporting students in making contributions*. In practice, revoicing involves re-saying a student's contribution to a classroom discussion. Forman and Ansell (2001) suggest that in classrooms where revoicing is used "…there is a greater tendency for students to provide the explanations…and for the teacher to repeat, expand, recast or translate student explanations for the rest of the class" (p. 119). In particular, the lesson study provided examples of linking between eliciting ideas and scaffolding the production of ideas through revoicing and extending.

PLANNING THE LESSON

The over-arching aim of learning experiences with whole numbers in the NSW elementary mathematics syllabus is that students develop a sense of the size of whole numbers and the role of place value in their representation. In New South Wales, students learn to subtract two, three and four digit numbers, typically using the decomposition method, in Years 3 and 4. Although students encounter one and two-digit subtraction in Years 1 and 2, only informal recording methods are used. The study lesson focusing on students communicating their reasoning orally was designed to draw upon familiar mathematical content. In this way, students working with the familiar content of two-digit subtraction should not have either their search for a pattern or their explanations impeded by additional challenges from the content.

As well as the topic of subtraction from the Number strand, the expectations of the Patterns and Algebra strand of the syllabus are that by Year 6, students can make basic generalizations about numbers and number relationships. However, describing the relationship between subtractions involving ordered two-digit numbers and the answer is well beyond the expectations of the elementary mathematics syllabus in NSW (Board of Studies NSW, 2002). Consequently, the planned level of abstraction referred to in the unit goal (Appendix A) is a very ambitious goal for the composite Year 5/6 class taking part in the lesson study.

The teacher's role in supporting communication

The patterns of communication intended by the Curious Subtraction lesson rely upon students making their thinking public. The teacher, in supporting students to make their contributions, used the design of the lesson to elicit their conjectures, propose next steps, and link together their justifications. Learning to communicate mathematical reasoning is fundamental to understanding mathematics. Palincsar and

Brown (1984) wrote that " …understanding is more likely to occur when a child is required to explain, elaborate, or defend his position to others; the burden of explanation is often the push needed to make him or her evaluate, integrate and elaborate knowledge in new ways."

The ways in which students seek to justify claims, convince their classmates and teacher, and participate in the collective development of publicly accepted mathematical knowledge contribute to mathematical argument. In a culture that expects student understanding, teaching mathematics is more than merely telling or showing students; teachers must enable students to create meanings through their own thinking and reasoning. Classroom argumentation needs opportunities to move from authority-based arguments (because the teacher says so or the text states this) to reasoning with mathematical backing (cf. Toulmin, 1969).

Ms. Ryan, the class teacher in the study lesson, focused her efforts on eliciting and linking students' ideas. She made sure that they had access to the conversation by ensuring a shared conception of the problem they were working on. Without common ground, students would not have the opportunity to consider and respond to others' ideas in meaningful ways. The lesson also demonstrated occasions where some students did not have access to the common ground.

THE LESSON: CURIOUS SUBTRACTION

The plan of the lesson is derived from the lesson video, *Curious subtraction* by Mr Hiroshi Tanaka (2008). At the start of the lesson, the teacher places cards showing the digits 0 to 9 on the board and selects two digit cards. A subtraction is then formed by arranging the digits in descending order making one two-digit number (the minuend) and reversing the order of the digits to form the subtrahend. By judiciously selecting the second of the two digits, the teacher is able to create the false proposition that any two-digit subtraction question formed from two distinct digits will appear to have the same answer. The students are then challenged to confirm or refute the proposition. This involves the whole class in generating and solving two-digit subtractions that are then shared on the board.

The students were then asked to provide a way of organising the answers to their investigation of finding the difference between a two-digit number and the number formed by reversing the order of the digits. After the subtractions were organised according to the answer (a structure prompted by the teacher), the teacher asks the students to have a look at the numbers and the answers, and tell her what they could see. The expectation is that the students can identify a pattern in the answers that they will describe and then seek to explain.

Eliciting students' ideas

One of the students (Jasmine) puts forward the observation that the answers are all multiples of nine. When the teacher states, "Is that right?" the student modifies her proposition to "most of them", apparently simply in response to having her answer questioned. Using Toulmin's argument model, Jasmine modifies her claim through the introduction of a modal qualifier rather than providing a warrant. The teacher confirms that the original proposition is correct, that is, she provides a warrant by linking the data to the claim, but follows up by emphasising that it is important to look even closer. The teacher focuses students' attention on the questions that result in an answer of nine.

Teacher: *What can you see about the answer, nine?… Vanessa?*

Vanessa: *When the two numbers are consecutive and when they are switched around, the answer is always nine.*

Teacher: *Is that right? Let's have a check. Consecutive. Consecutive. Consecutive. … What does consecutive mean?*

The teacher checks that the discussion is creating shared meaning by testing whether the term "consecutive" is part of *taken as shared meaning* for the class.

Figure 1. Explaining the relationship between the digits and the answer

The search for a relationship between the digits in the subtractions and the answers moves on to look at those subtractions with an answer of eighteen. The teacher continued to scaffold the production of student ideas and to link responses. The push towards effective communication of mathematics is evident when Ben says that the numbers producing an answer of 18 "go by twos" and the teacher says, "Yes, but if we were explaining it to someone…" The teacher then provides an explanatory scaffold: "For an answer to be 18 the digits…". Eventually, the teacher revoices the student's explanation that for the answer to be 18, the digits have to have a difference of two, and returns to the earlier observation of the relationship to answers of nine. Moving the explanation from a focus on consecutive numbers to the difference between the digits is an important pedagogical move. Expressing the relationship between the digits and the answer is easier if the same sentence structure is used.

Jasmine: *Well, with all of them, if it's a difference of two, then you times two by nine with all of them*

So, like, the difference of two goes with eighteen, and with 63 it's a difference of seven, which goes with nine, equals 63.

Teacher: *OK. So what Jasmine is saying is that it's the difference…*

Jasmine: *Timesed by nine.*

Teacher: *Does that work for everyone of them?*

The teacher tests out the conjecture using those questions with an answer of seventy-two. The students confirm the conjecture and work to refine the statement of the conjecture. Having established the conjecture, the teacher starts to treat it as a rule and begins to determine if students can apply the rule. The next part of the lesson is designed to see if students can explain why the result occurs. The teacher returns to eliciting students' ideas and the struggle to describe the result in terms of place value. Ms Ryan encourages attempts and reinforces helpful steps towards an explanation. A few of the students appear to come close to explaining why the result occurs when the digits are one apart. However, the goal of describing why there is a relationship between the difference in the digits and the corresponding multiple of nine is not reached.

To help to distil the reasoning, the teacher asks students to reverse the problem. That is, given an answer of 45, the students were asked to work in pairs to find all of the pairs of digits that result in 45 as the answer to the subtraction. The discussion in pairs indicated a range of strategies, from those who struggled with the task through to those who adopted a systematic approach and even tried to extend the problem to three digit numbers.

REFINING THE LESSON

Although the lesson was effective at engaging students in communicating their reasoning, coordinated through the efforts of the teacher, refinements need to be made to come closer to the lesson goal. One simple change to the lesson would be to achieve better organization of the answers to the problem on the board. This improved organization of board work would also allow the teacher to offer a notation to focus student attention on the difference between the digits, and the result of the subtraction. The impact on the answers to the subtraction questions due to the interchange of the digits between the units and tens position would be strengthened through the introduction of a simple notation (see for example Figure 2).

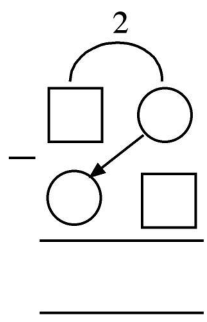


Figure 2 Introducing a tracking notation.

Keeping track of the movement of the digits and the difference between the digits needs to be supported through a non-opaque symbolisation of the problem. Bruner (1973) describes a "transparent" use of symbols as one where actions are guided by reasoning about the entities to which the inscriptions are assumed to refer. That is, in learning to use appropriate language and representations to formulate and express mathematical ideas in written, oral and diagrammatic form, greater emphasis needs to be given to the link between oral (including gesture) and diagrammatic notations. Indeed, the diagrammatic notations can at times build on students' gestures. At one point in the study lesson, a student uses an "arching" gesture when describing the relationship between the digits. This type of gesture (Figure 3) provides an opportunity to introduce a linking notation identifying the difference between the digits.

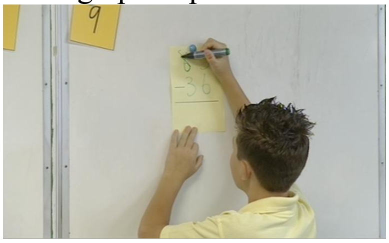
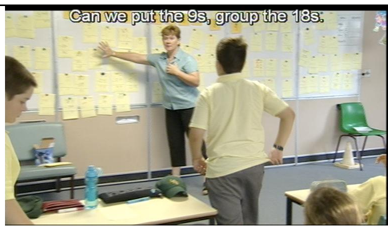
Figure 3 An opportunity to link gesture to notation

If a student does not provide a supporting gesture in attempted explanations, the teacher can suggest it in revoicing explanations. A good notation or system of representation is central to algebraic modes of thinking and needs to be introduced in the study lesson. A good notation might be similar to what has been described as a *conceptual* diagram (Nakahara, 2007). Effective notations reduce the cognitive load associated with trying to communicate complex quantitative relationships. The notation system should be developed as an extension of the students' attempts to generalise.

In developing mathematical communication, students need opportunities to use appropriate language and representations to formulate and express their mathematical ideas. These representations can at times build on the role of gesture in mathematical thinking, as gestures can act as a tool for thinking and lighten working memory load when explaining mathematics (Goldin-Meadow, 1999). However, a good representational system must form part of the mathematical communication in developing students' explanations of why a mathematical relationship holds (Kaput, Noss, & Hoyles, 2002). The revised lesson plan will emphasise the introduction of a good notation to assist communicating the reasoning for *Curious subtraction*.

APPENDIX A: UNDERSTANDING CURIOUS SUBTRACTION (GRADE 5-6)

Goal: To have students test out a conjecture, find a pattern involving the digits in a subtraction and the answer, communicate the pattern and try to find a reason for the pattern.

	OUTLINE	**COMMENTS**
Introducing the proposition	Place cards with the digits 0 to 9 on the board and invite students to select two of the digits. Ask which two-digit numbers can be formed using the two selected digits. Create a subtraction question by using one digit in the tens place and the other in the units position and reversing the order of the digits to produce the second number. The teacher's selection of the second digit intentionally keeps the difference between the two digits the same for each question, resulting in the (false) proposition that the answer will always be the same. Have the students generate examples to confirm or disprove the proposition.	Setting up the problem.  Establishing the conjecture. 
Providing structure	Ask the students to suggest a way of arranging the questions to make the search for a pattern easier. Encourage the students to search for a pattern. Do not stop at the first pattern identified. Check that any pattern that a student identifies is described in a way that supports other students understanding what is meant. If needed, focus on one group of questions all with the same answer, say, nine. Link and sequence students' attention to parts of the data as necessary. Test out students' claims using the available data on the board.	 Students initially identified that the answers are all multiples of nine. 
Communicating reasoning	Ask the class to describe the relationship between the digits in the question and the answer using mathematical language. Have students test out their understanding by describing the relationship in their own words. Ask the students to work in pairs to find all of the two digit subtraction questions of the type being studied that have an answer of 45. Look at the strategies students use to find all of the answers. Check if the students are able to describe how they know they have all of the answers.	Students work in pairs to look at finding all solutions to 45. 

References

Board of Studies NSW. (2002). *Mathematics K-6 syllabus*. Sydney: Board of Studies NSW. (http://k6.boardofstudies.nsw.edu.au/go/mathematics)

Bruner, J. (1973). *Beyond the information given.* New York: Norton & Co.

Cobb, P., & Bauersfeld, H. (Eds.). (1995). *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 271–291). Hillsdale, NJ: Lawrence Erlbaum Associates.

Forman, E. & Ansell, E. (2001). The multiple voices of a mathematics classroom community. *Educational Studies in Mathematics, 46,* 114–142.

Goldin-Meadow, S. (1999). The role of gesture in communication and thinking. *Trends in Cognitive Sciences, 3,* 419–429.

Kaput, J., Noss, R., & Hoyles, C. (2002). Developing new notations for a learnable mathematics in the computational era. In L. English (Ed.) *Handbook of International Research in Mathematics Education*. pp. 51–75. Mahwah, NJ: Lawrence Erlbaum.

Mehan, H. (1979). *Learning lessons.* Cambridge, MA: Harvard University Press.

Nakahara, T. (2007). *Cultivating mathematical thing through representation: Utilizing the representational system* [PDF document]. Retrieved from http://www.criced.tsukuba.ac.jp/math/apec/apec2008/index_en.php

Staples, M. (2007). Supporting whole-class collaborative inquiry in a secondary mathematics classroom. *Cognition and Instruction, 25*, 161–217.

Sinclair, J & Coulthard, M. (1975). *Towards an Analysis of Discourse: The English used by Teachers and Pupils.* London: Oxford University Press.

Tanaka, H. (video retrieved 2008). *Curious subtraction.* Retrieved from http://www.globaledresources.com/resources/curious-subtraction.html

Toulmin, S. (1969). *The Uses of Argument*, Cambridge, England: Cambridge University Press.

Wood, T. (1994). Patterns of interaction and the culture of mathematics classrooms. In S. Lerman (Ed.), *The culture of the mathematics classroom* (pp. 149–168). Dordrecht: Kluwer.

Wood, T., Williams, G., & McNeal, B. (2006). Children's mathematical thinking in different classroom cultures. *Journal for Research in Mathematics Education, 37*, 222-255.